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New modular invariance in the $\mathcal{N}=1^*$ theory, operator mixings and supergravity singularities

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Abstract: We discuss the mass-deformed $\mathcal{N}=4$ SU(N) supersymmetric Yang-Mills theory (also known as the $\mathcal{N}=1^*$ theory). We analyze how the correlation functions of this theory transform under S-duality, and which correlation functions depend holomorphically on the complexified gauge coupling τ. We provide exact modular-covariant expressions for the vacuum expectation values of chiral operators in the massive vacua of the $\mathcal{N}=1^*$ theory. We exhibit a novel modular symmetry of the chiral sector of the theory in each vacuum, which acts on the coupling $\tilde{\tau}=(p\tau+k)/q$, where p, k and q are integers which label the different vacua. In the strong coupling limit, we compare our results to the results of Polchinski and Strassler in the string theory dual of this theory, and find non-trivial agreement after operator mixings are taken into account. In particular we find that their results are consistent with the predicted modular symmetry in $\tilde{\tau}$. Our results imply that certain singularities found in solutions to five-dimensional gauged supergravity should not be resolvable in string theory, since there are no field theory vacua with corresponding vacuum expectation values in the large-N limit.

Keywords: Nonperturbative Effects, 1/N Expansion, AdS/CFT Correspondance, Supersymmetry and Duality.

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Contents

1. Introduction and summary

The AdS/CFT correspondence [1, 2, 3] (see [4] for a review) is conjectured to be an exact duality between certain field theories and certain compactifications of string/M-theory. As originally stated the correspondence applied only to conformal field theories, but it can easily be generalized also to relevant deformations of the conformal field theories, which are realized as solutions of string/M-theory with particular boundary conditions depending on the deformation parameter. The AdS/CFT correspondence relates weakly coupled field theories to highly curved backgrounds of string/M-theory, and vice versa, a fact which makes it quite difficult to test. Most of the tests of the correspondence so far are either qualitative in nature, or they involve quantities that do not depend on the coupling. While the AdS/CFT correspondence has taught us a lot about both field theory and string theory in regions of parameter space that were previously completely inaccessible (such as field theories with large $g_{\mathrm{YM}}^2 N$), it has not yet been possible to use it to learn about more traditional theories such as gauge theories which are weakly coupled at some energy scale, or string theory compactifications including regions of small curvature. One of the goals of this paper will be to try to use field theory and the AdS/CFT correspondence to learn about string theory backgrounds that include regions of small curvature (where supergravity is valid) but where supergravity seems to break down at some singularities, and to check whether the resulting singularities can be resolved in string theory or not.

In this paper we will study the SU(N) $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in four dimensions, deformed by a superpotential which gives a mass to the adjoint scalars and to three of the four adjoint fermions and breaks the supersymmetry to $\mathcal{N} = 1$; this theory was dubbed $\mathcal{N} = 1^*$ in [5]. The deformed theory has a finite (but large for large N) number of possible vacuum states preserving supersymmetry, some of which have a mass gap while others do not. On the string theory side of the AdS/CFT correspondence, these theories were studied in [6] and in [5]. We will study this deformation by field theory methods, using the results of [7, 8] which allow exact computations of vacuum expectation values (VEVs) of various chiral operators in these theories, in each vacuum and for every value of the coupling, as well as computations of the tensions of BPS-saturated domain walls interpolating between pairs of vacua. For large N and strong 't Hooft coupling these theories are dual, by the AdS/CFT correspondence, to string theory backgrounds that asymptote to $AdS_5 \times S^5$ with a large radius of curvature. We will compare the field theory results to results found using supergravity in this limit.

The authors of [5] looked for solutions of ten-dimensional type-IIB supergravity (SUGRA) with possible 5-brane sources (but without any other singularities) that would correspond to vacua of the mass-deformed theory. They found a large number of solutions for which the supergravity (+ 5-brane sources) approximation could be trusted (for a large range of possible values of the coupling), which are approximate string theory backgrounds corresponding to many of the field theory vacua. The solutions found in [5] are isolated (up to the choice of the asymptotic string coupling) and were identified with particular vacua of the field theory. The authors of [5] were also able to construct supergravity solutions corresponding to BPS domain walls interpolating between distinct vacua. The tensions of these objects were found to agree with the field theory predictions of [8]. In the following we will perform a similar comparison between gauge theory and SUGRA for the chiral condensates. In particular, using the solutions of [5] we can compute the VEVs of various operators at strong coupling and compare with the exact field theory results, allowing a quantitative test of the AdS/CFT correspondence at strong coupling. The comparison of condensates is complicated by the fact that the symmetries allow for non-trivial mixings among the chiral operators after the mass deformation, so there is no unique way to define the field theory operators corresponding to supergravity fields. The resolution of this ambiguity involves a novel modular symmetry of the $\mathcal{N} = 1^*$ theory which we now discuss.

As reviewed below, the $\mathcal{N} = 1^*$ SU(N) theory has a total of $\sum_{d|N} d$ massive vacua, labeled by three non-negative integers, p, q, and $k < q$, with $pq = N$. These vacua are permuted by SL$(2, \mathbb{Z})$ modular transformations acting on the complexified coupling constant [9] $\tau = 4\pi i/g^2_{\mathrm{YM}} + \theta_{\mathrm{YM}}/2\pi$. Thus, the S-duality of the underlying $\mathcal{N} = 4$ theory is not a symmetry of the $\mathcal{N} = 1^*$ theory in a given vacuum, but rather relates the physics in distinct vacua at different values of τ. However, it turns out that a sector of the theory in each vacuum is actually invariant under a different modular group which acts on the coupling $\tilde{\tau} = (p\tau + k)/q$. As we review below, this symmetry, which we will refer to as $\tilde{\text{S}}$-duality, has a simple explanation in terms of the hyperelliptic curves given by Donagi and Witten [9] which govern the Coulomb branch of the corresponding theory with eight supercharges. This connection suggests that $\tilde{\text{S}}$-duality is probably only a symmetry of the chiral sector of the theory, which is controlled by the corresponding curve. We find that there is a unique definition for each chiral operator which transforms with definite weight under $\tilde{\text{S}}$-duality. In some vacua, $\tilde{\text{S}}$-duality transformations relate two different regimes where the SUGRA solutions of [5] are reliable. These cases yield non-trivial agreement between gauge theory and SUGRA. More generally, it is notable that in the SUGRA approximation the solutions of [5] only depend on $\tilde{\tau}$ and not on p, q, k and τ separately. This exclusive dependence on $\tilde{\tau}$ is itself a necessary condition for $\tilde{\text{S}}$-duality.

The AdS/CFT dual of the $\mathcal{N} = 1^*$ theory has also been studied by different methods in [6]. These authors looked for solutions of the five-dimensional $\mathcal{N} = 8$ gauged supergravity that had the correct asymptotic behavior to describe the mass-deformed $\mathcal{N} = 4$ SYM theory, and that allowed for VEVs of some (but not all) of the operators of the theory. They found a class of solutions with a continuous VEV for the supergravity mode corresponding to a linear combination of the gluino condensate and other fields. These solutions all develop a naked singularity at some value of the radial coordinate (the solutions are found by integrating the equations of motion in the radial direction, starting from particular boundary conditions set by the deformation parameters and the operator VEVs). Assuming that $d = 5$, $\mathcal{N} = 8$ supergravity is a consistent truncation of type-IIB supergravity on $AdS_5 \times S^5$, the solutions of [6] may be lifted to solutions of type-IIB supergravity, which presumably also involve a naked singularity. Due to the presence of the singularity it is not clear if these solutions correspond to consistent string theory backgrounds (which would resolve the singularity in some way) or not. Since the solutions of [6] feature a continuous VEV for the gluino condensate, while the field theory has (for finite N) just a finite number of possible vacua, it is clear that for finite N the solutions of [6] cannot all be physical. However, since the number of vacua grows rapidly in the large-N limit (as $e^{\sqrt{N}}$), it is possible for the discrete vacua to look like a continuum in this limit, so that every solution found in [6] might be the limit of some series of allowed vacua. If this were true, it would mean that the singularities found in [6] could all be resolved by string theory (since we would have a consistent string theory solution corresponding to every vacuum of the field theory). Our field theory analysis will lead us to a different conclusion. It seems that only one of the solutions of [6] corresponds to the large-N limit of a series of vacua of the field theory, while the other solutions do not seem to correspond to such a limit, so presumably the singularities which appear in them should not be resolvable in string theory.

We will not be able to prove this rigorously since we have field theory results only for some of the vacua of the theory, namely the vacua with a mass gap and a small number of massless vacua. However, we will be able to show that none of these vacua converge to generic solutions of the type found in [6], and we consider it unlikely that other series of vacua would have a good large-N limit (though it is not impossible).

At first sight, one would think that there should be a correspondence between the solutions found in [5] and in [6]. However, generally this is not the case because the solutions of [5] include 5-brane sources while those of [6] do not. In many of the vacua discussed in [5], there are 5-branes sitting at a radial position of order $\sqrt{N}$ (in units of the asymptotic AdS radius of curvature), which significantly change the solutions, so there is no sense in which these solutions of [5] are related to the solutions of [6]. In this case it seems that instead of resolving the singularities found by [6], string theory replaces them by completely different spaces which do not resemble the solutions which become singular. For some particular vacua, all the 5-branes of [5] sit at a finite radial position in the large-N limit, and then the solutions of [5] should be similar to those of [6], at least at large radial positions where the effects of the 5-branes are small (and for particular choices of parameters leading to the VEVs analyzed in [6]). Thus, in these cases one might say that string theory resolves the singularities of [6] by replacing them with 5-branes. Unfortunately, as one tries to "push" the 5-branes to smaller radial positions (where the solutions of [6] become singular) the approximations used in [5] break down, so more work is needed to understand exactly how the singularities are resolved by string theory.

We will begin in section 2 by reviewing the spectrum of chiral operators of $\mathcal{N}=4$ SYM and their modular transformation properties under the $\mathrm{SL}(2,\mathbb{Z})$ electric-magnetic duality of the theory, which will be useful for our analysis. In section 3 we introduce the mass-deformed theory and discuss which correlation functions in this theory should have a holomorphic dependence on the gauge coupling τ. In section 4 we review the field theory results of [7, 8] for the domain wall tensions and also obtain expressions for operator VEVs in different vacua, and we verify their consistency with modular invariance. In section 5 we discuss operator mixing and the new modular symmetry of each vacuum state described above, and show how to define chiral operators which are covariant under this symmetry. In section 6 we analyze the VEVs in supergravity limits, and show that the supergravity results of [5] agree with the field theory results for the operator VEVs if we use the $\tilde{\mathrm{S}}$-covariant definition of the operators. In sections 7 and 8 we look for vacua of the field theory which could correspond to the solutions described by [6], and show that some of the singularities found in [6] should be resolvable in string theory (presumably by replacing them by branes as in [5]) while others should not be resolvable. We also comment on the possible relation of this result to other singular backgrounds which have recently been discussed in the context of solutions to the cosmological constant problem.

JHEP06(2000)026

2. Modular properties and normalizations of $\mathcal{N}=4$ operators

We begin by reviewing some basic facts regarding the $\mathcal{N}=4$ theory with SU(N) gauge group in four dimensions. Schematically, the lagrangian for this theory is

$$\mathcal{L} = N\,\frac{1}{g_{\rm YM}^2 N}\ {\rm tr}\left(-\frac{1}{4}F_{\mu\nu}^2 + \mathcal{D}^\mu\phi^I\,\mathcal{D}_\mu\phi^I + \left[\phi^I,\phi^J\right]^2 + \text{fermions}\right) + \\ + \frac{\theta}{8\pi^2}\ {\rm tr}(F\wedge F)\,. \tag{2.1}$$

The ϕ^I ($I=1,\ldots,6$) are SU(N)-adjoint scalars in the **6** representation of the global SO(6) R-symmetry group. This theory possesses superconformal invariance and is also believed to be invariant under SL$(2,\mathbb{Z})$ transformations which act on $\tau \equiv 4\pi i/g_{\rm YM}^2 + \theta/2\pi$ as $\tau\to\tau' = (a\tau+b)/(c\tau+d)$. Consequently, the theory with parameter τ describes the same physics as the theory with parameter τ' (on $\mathbb{R}^4$).

The $\mathcal{N}=4$ theory can also be described as an $\mathcal{N}=1$ theory with 3 adjoint chiral multiplets Φ_i and a superpotential proportional to ${\rm tr}(\Phi_1[\Phi_2,\Phi_3])$. The scalar components of these superfields may be written in terms of the real fields $\{\phi^I\}$ as $\Phi_i = (\phi^i + i\phi^{i+3})/\sqrt{2}$ (we use the same symbol Φ_i for the $\mathcal{N}=1$ chiral superfield and for its lowest component).

Following Intriligator [10], it is convenient to define the lowest components of the chiral primary superfields of the $\mathcal{N}=4$ superconformal algebra to be

$$\mathcal{O}_p^{I_1 I_2\cdots I_p} \equiv N\,(g_{\rm YM}^2 N)^{-p/2}\ {\rm tr}\left(\phi^{\{I_1}\,\phi^{I_2}\cdots\phi^{I_p\}}\right). \tag{2.2}$$

Here the I_j are SO(6) indices contracted to form a symmetric traceless product of the p **6**'s, corresponding to representations of weight $(0,p,0)$ of the SU(4) $\simeq$ SO(6) R-symmetry group. In the normalization of (2.2) the chiral primary operators $\mathcal{O}_p$ are SL$(2,\mathbb{Z})$ invariant.[1] Furthermore, given the normalization (2.2), in the large-N limit all correlation functions of $\mathcal{O}_p$'s and their descendants are proportional to N^2, and they also have a smooth limit as $\lambda\equiv g_{\rm YM}^2 N\to 0$. This is the appropriate leading behavior for correlators in the large-N limit, implying that these operators in the $\mathcal{N}=4$ theory can be directly identified with type-IIB supergravity fields without introducing any additional factors of N.

Since the operators $\mathcal{O}_p$ are SL$(2,\mathbb{Z})$ invariant, the fields ${\rm tr}(\phi^p)$ (omitting the SU$(4)_R$ indices for brevity) transform under SL$(2,\mathbb{Z})$ as modular forms with modular weight[2] $(p/2,p/2)$ (recall that Im(τ) is a modular form of weight $(-1,-1)$). The $\mathcal{N}=4$ supercharges Q^A_α and $\bar{Q}_{A\dot\alpha}$ can also be thought of effectively as modular forms with weight $(1/4,-1/4)$ and $(-1/4,1/4)$, respectively [10]. Therefore, the descendant fields obtained by acting on $\mathcal{O}_p$ as defined in (2.2) with k powers of Q and l powers of $\bar{Q}$ transform as modular forms of weight $((k-l)/4,-(k-l)/4)$.

[1] With the kinetic terms normalized as in (2.1), this is clearly true for instance in the case of the U(1) theory which is free.

[2] $\mathcal{O}$ is a modular form of weight $(w,\bar{w})$ if when $\tau\to(a\tau+b)/(c\tau+d)$, $\mathcal{O}\to(c\tau+d)^w\,(c\bar\tau+d)^{\bar w}\,\mathcal{O}$.

JHEP06(2000)026

In what follows we will be primarily interested in the VEVs for certain chiral operators (in the $\mathcal{N}=1$ sense) in the mass-deformed version of $\mathcal{N}=4$ SYM. The operators of interest are some of the $\mathcal{N}=4$ operators $\mathcal{O}_2$ (with a special choice of $\mathrm{SU}(4)_R$ indices so that $\mathcal{O}_2=\mathcal{O}_2^{(i)}\equiv \mathrm{tr}(\Phi_i^2)/g_{\rm YM}^2$), and a particular scalar descendant of this operator obtained by acting on it with two supercharges of the same chirality. We will schematically denote this descendant as $Q^2\,\mathcal{O}_2$, which is shorthand for the nested commutator $\{Q^A_\alpha,[Q^B_\beta,\mathcal{O}_2^{(i)}]\}\epsilon_{\alpha\beta}$. This operator will be discussed in more detail below. The operator $\mathcal{O}_2^{(i)}$ is a component of $\mathrm{tr}(\phi^{\{I}\phi^{J\}})/g_{\rm YM}^2$, which is in the $\mathbf{20'}$ representation of $\mathrm{SU}(4)_R$ and is $\mathrm{SL}(2,\mathbb{Z})$ invariant, while $Q^2\,\mathcal{O}_2$ is a complex operator in the $\mathbf{10}$ representation of $\mathrm{SU}(4)_R$ and is a modular form of weight $(1/2,-1/2)$. Note that $\mathrm{tr}(\Phi_i^2)$ itself is a modular form of weight $(1,1)$.

Naively (as reviewed in [4]) the action of the two supercharges on $\mathcal{O}_2$ gives rise to two types of terms. When the supercharges each act on a distinct term in the bilinear, a symmetric product of two fermions ψ of the same chirality is obtained. The action of both supercharges on the same scalar gives (choosing an $\mathcal{N}=1$ subalgebra) F and D terms of the form $[\phi^I,\phi^J]$. Thus, classically one expects that the various components of $Q^2\,\mathcal{O}_2$ in the $\mathcal{N}=4$ theory must be of the schematic form $\sim \mathrm{tr}(\psi\,\psi+\phi^I[\phi^J,\phi^K])$, where the indices on the fermions and scalars are contracted to give an operator in the $\mathbf{10}$ representation of $\mathrm{SU}(4)_R$ (if we choose Q to transform in the $\bar{\mathbf{4}}$ representation).

Specifically, choosing an $\mathcal{N}=1$ subalgebra and denoting the gluino by ψ_4, for a particular choice of $\mathrm{SU}(4)$ indices the action of two supercharges Q^A_α on $\mathcal{O}_2^{(i)}$ yields

$$Q^2\,\mathcal{O}_2\Big|_{\rm classical}\equiv \frac{1}{g_{\rm YM}^2}\,Q^2\,\mathrm{tr}(\Phi_i^2)=\frac{1}{g_{\rm YM}^2}\;\mathrm{tr}\left(\psi_4\,\psi_4+2\,\Phi_1[\Phi_2,\Phi_3]\right). \tag{2.3}$$

As indicated, this expression is valid only classically. The double action of the supercharges on the composite operator $\mathcal{O}_2^{(i)}$ also gives rise to an anomalous term which shows up at one loop. In $\mathcal{N}=1$ supersymmetric gauge theories in general, this term is known as the Konishi anomaly [11, 12], and in the theory with adjoint fermions it is proportional to $g_{\rm YM}^2\,N\,\mathrm{tr}(\psi\psi)$ (with a coefficient of order one).

The existence of this term can be shown by regulating operator products involving $Q^2\,\mathcal{O}_2$ via a convenient regularization scheme, such as Pauli-Villars. The finite contribution from the Pauli-Villars regulator fields then leads to the Konishi anomaly, as in [11]. Therefore, we find that $Q^2\,\mathcal{O}_2$ is given by

$$Q^2\,\mathcal{O}_2=\frac{1}{g_{\rm YM}^2}\;\mathrm{tr}\left(\psi_4\,\psi_4+2\,\Phi_1[\Phi_2,\Phi_3]+g_{\rm YM}^2\,NK\right), \tag{2.4}$$

where K represents the Konishi anomaly. Note the relative factor of 2 between the first and second terms, which will be significant below. In the weak coupling limit $K\sim\mathrm{tr}(\psi_4\,\psi_4)$. Although the Konishi anomaly may be argued to be 1-loop exact in certain cases, we do not discount the possibility of further perturbative and non-perturbative contributions. In fact, the exact form of the anomaly will not concern us. What will turn out to be important is that the same anomaly also appears in $\bar{Q}^2_{\mathcal{N}=1}\,\mathrm{tr}(\bar{\Phi}_i\,\Phi_i)$, where $Q_{\mathcal{N}=1}$ and $\bar{Q}_{\mathcal{N}=1}$ represent the unbroken supercharges of the mass-deformed $\mathcal{N}=4$ theory.

3. $\mathrm{SL}(2,\mathbb{Z})$ and holomorphy in the $\mathcal{N}=1^*$ theory

The mass-deformed $\mathcal{N}=4$ theory (henceforth referred to as the $\mathcal{N}=1^*$ theory as in [5]) is obtained by introducing explicit mass terms for the adjoint chiral multiplets, such that the superpotential is

$$W = -\frac{\tau}{16\,\pi\,i}\,\mathrm{tr}(W_\alpha^2) + \frac{1}{g^2_{\mathrm{YM}}}\,\mathrm{tr}\left(\Phi_1[\Phi_2,\Phi_3] + m_1\,\Phi_1^2 + m_2\,\Phi_2^2 + m_3\,\Phi_3^2\right). \qquad (3.1)$$

The theory has $\mathcal{N}=2$ SUSY in the special case where one of the three masses vanishes and the other two are equal. We will refer to this as the $\mathcal{N}=2^*$ theory. Although no special holomorphy properties of the $\mathcal{N}=4$ theory or its mass deformation are apparent from (2.1) and (3.1), it can be argued in the $\mathcal{N}=1$ and $\mathcal{N}=2$ languages that certain correlation functions in these theories depend holomorphically on τ and on the mass parameters. These holomorphy properties are not evident in (2.1) and (3.1) since the gauge kinetic term (and terms related to it by $\mathcal{N}=1$ or $\mathcal{N}=2$ supersymmetry), the kinetic terms for the chiral multiplets and the superpotential are all proportional to $1/g^2_{\mathrm{YM}} \sim \mathrm{Im}(\tau)$. Thus, naively one would not expect correlators in the theory to have holomorphic dependence on τ.

To get a holomorphic dependence on τ we need to normalize the fields so that $\mathrm{Im}(\tau)$ does not appear in the superpotential. Since the $\mathcal{N}=4$ superpotential is proportional to $\mathrm{tr}(\Phi_1[\Phi_2,\Phi_3])/g^2_{\mathrm{YM}}$, and we want to get a superpotential proportional to N but not to $1/g^2_{\mathrm{YM}}$, we need to work in terms of rescaled fields $\tilde{\Phi}_i \equiv \Phi_i/\lambda^{1/3}$. The kinetic terms of the $\tilde{\Phi}_i$ have an ugly dependence on g_{YM}, but the superpotential is now simply

$$W = \mathrm{tr}\left[-\frac{\tau}{16\,\pi\,i}\,W_\alpha^2 + N\left(\tilde{\Phi}_1\left[\tilde{\Phi}_2,\tilde{\Phi}_3\right] + \tilde{m}_1\,\tilde{\Phi}_1^2 + \tilde{m}_2\,\tilde{\Phi}_2^2 + \tilde{m}_3\tilde{\Phi}_3^2\right)\right], \qquad (3.2)$$

where $\tilde{m}_i \equiv m_i/\lambda^{1/3}$ (note that due to the non-standard normalization of the kinetic terms, the physical masses of the chiral superfields are still given by m_i and not by $\tilde{m}_i$). General arguments (such as promoting τ in the superpotential to a chiral multiplet whose lowest component has a VEV τ) can now be used to show that correlation functions of the fields $\tilde{\Phi}_i$ and W_α should be holomorphic in $\tilde{m}$ and τ.

In summary, while the natural fields and physical parameters to use in these theories are Φ_i and m_i, respectively, the holomorphy properties become clear when the theory is rewritten in terms of the fields $\tilde{\Phi}_i$ and parameters $\tilde{m}_i$. The modular properties of the operators in the $\mathcal{N}=1^*$ theory can now be easily deduced by examining the relationship between the two sets of fields and parameters.

As in (3.1), the mass deformation naturally appears with a coefficient $1/g^2_{\mathrm{YM}}$ like all the other terms in the $\mathcal{N}=4$ lagrangian. The parameter m_i therefore couples (at leading order in the mass) to a field of the form $Q^2\,\mathrm{tr}(\Phi_i^2)/g_{\rm YM}^2$, which is a component of $Q^2\,\mathcal{O}_2$. As described above, this operator has the appropriate normalization for comparing with supergravity and has modular weight $(1/2,-1/2)$. Therefore the m_i must naturally have modular weight $(-1/2,1/2)$ to ensure modular invariance (i.e. we can assign modular transformations with this weight to the m_i in order to preserve modular invariance after the mass deformation). The m_i are related to the natural holomorphic parameters by $\tilde{m}_i = m_i/\lambda^{1/3} \propto m_i(\mathrm{Im}(\tau)/N)^{1/3}$, so $\tilde{m}_i$ has modular weight $(-5/6,1/6)$.

From the above analysis, gauge invariant operators of the form $u_p \simeq \mathrm{tr}(\Phi^p)$ with no additional dependence on $g_{\rm YM}$ transform as modular forms of weight $(p/2,p/2)$ in the $\mathcal{N}=4$ theory [10]. These fields are related to the holomorphic coordinates $\tilde{u}_p \simeq \mathrm{tr}(\tilde{\Phi}^p)$ by $\tilde{u}_p = u_p/\lambda^{p/3} \propto (\mathrm{Im}(\tau)/N)^{p/3}\,u_p$. Thus, the natural holomorphic fields $\tilde{u}_p$ have modular weight $(p/6,p/6)$. Holomorphy now determines relations like $\tilde{u} \sim \tilde{m}^2 e_i(\tau)$ for the location of the massive vacua of the mass-deformed SU(2) $\mathcal{N}=4$ theory [13] in which both sides have the same modular weight $((1/3,1/3) = 2(-5/6,1/6)+(2,0))$; these lead to similar relations $u \sim m^2 e_i(\tau)$ for the more standard fields after rescaling by appropriate powers of λ.

Before moving on to a detailed discussion of the condensates in this theory we would like to clarify the relation of the chiral fields u_p above, with modular weight $(p/2,p/2)$, to the gauge-invariant parameters with modular weight $(p,0)$ appearing in the Donagi-Witten curves [9] for the $\mathcal{N}=4$ theory with an $\mathcal{N}=2$-preserving adjoint mass term. We note that the variables x, y and t in the Donagi-Witten curves $F(t,x,y)=0$ can be rescaled by powers of the masses such that the curves only depend on u_p/m^p. Choosing mass parameters that transform as $(-1/2,1/2)$ forms as above then implies that the quantities u_p/m^p on which the curves depend have weight $(p,0)$. This agrees with the conventions of Donagi and Witten, who use modular-invariant masses and gauge-invariant fields u_p transforming as $(p,0)$ forms so that the dimensionless fields u_p/m^p have weights $(p,0)$. A related issue is that while one would expect the holomorphic fields $\tilde{u}$ and $\tilde{m}$ to have holomorphic modular weights, we found that they do not. However, if we assign R-charges to the masses so that the mass-deformed theory has a $\mathrm{U}(1)_R$ symmetry with charge $2/3$ for $\tilde{\Phi}$ and $\tilde{m}$, then the anti-holomorphic weights of all the holomorphic fields are just $1/4$ of their R-charge. The expectation value of any combination of $\tilde{u}_p$'s is given by a polynomial in the masses with the same R-charge times some function of τ, and this function will always have a purely holomorphic modular transformation as expected.

4. Vacua and condensates in the $\mathcal{N}=1^*$ theory

Having identified the relation between the holomorphic operators and the operators with good modular transformation properties, we can now compute VEVs for the latter in the $\mathcal{N}=1^*$ vacua using the known results for the holomorphic operators.

JHEP06(2000)026

Up to possible operator redefinitions which will be discussed below, the condensates of the holomorphic operators $\mathrm{tr}(\tilde{\Phi}_i^2)$ and $\mathrm{tr}(W_\alpha^2)$ can be determined from [7] where an exact superpotential for the mass-deformed $\mathcal{N}=4$ theory was obtained. Since this superpotential is holomorphic, the results above imply that it should be written in terms of the parameters $\tilde{m}_i$. The effective superpotential of the SU(N) $\mathcal{N}=1^*$ theory on $\mathbb{R}^3\times S^1$ is of the form [7]

$$W = N\,\tilde{m}_1\,\tilde{m}_2\,\tilde{m}_3 \sum_{1\le a<b\le N} \mathcal{P}(X_a - X_b|\tau)\,, \tag{4.1}$$

where the X_a ($a=1,\ldots,N;\sum_a X_a=0$) are the chiral superfields arising from the gauge field after compactifying on a circle at a generic point on the Coulomb branch. From this superpotential we can read off the four-dimensional superpotential in every vacuum of the theory (corresponding to an extremum of (4.1)). The massive vacua of the theory are labeled [14, 9] by possible factorizations of N as $N=p\,q$ and by an integer $k=0,\ldots,q-1$, and the value of the superpotential in these vacua turns out to be[3]

$$W = \frac{N^3}{24}\,\tilde{m}_1\,\tilde{m}_2\,\tilde{m}_3\left[E_2(\tau) - \frac{p}{q}\,E_2\left(\frac{p}{q}\tau + \frac{k}{q}\right)\right]. \tag{4.2}$$

The second Eisenstein series $E_2(\tau)$ [15] is the holomorphic function which is closest to being a modular form of weight two; its modular transformation properties are $E_2(\tau+1)=E_2(\tau)$ and $E_2(-1/\tau)=\tau^2\,E_2(\tau)-6i\,\tau/\pi$.

It should be noted that within field theory there is actually an ambiguity in the superpotentials (4.1) and (4.2), corresponding to a possible additive holomorphic contribution [8] $A(\tau,N)$, with a weak coupling expansion of the form $\alpha_0+\sum_k \alpha_k\,e^{2\pi i k\tau}$. As the shift is independent of the chiral superfields X_a, it is the same in each vacuum of the theory. In particular, the ambiguity corresponds to the addition of an arbitrary holomorphic function of τ, which is independent of p, q and k, to the superpotential (4.2). However, it is worth noting that any such shift will spoil the modular transformation properties of W (apriori there is no reason for W to have nice modular transformation properties, but as discussed below $\partial W/\partial\tilde{m}_i$ is naturally a modular form, and in the $\mathcal{N}=1^*$ theory $\partial W/\partial\tilde{m}_i = W/\tilde{m}_i$). Indeed, to preserve the modular properties of (4.2), the function A would need to be a holomorphic modular form of weight two. Here we are assuming that the superpotential does not have any unphysical singularities in the interior of the fundamental domain of SL$(2,\mathbb{Z})$. As no such forms exist, we see that (4.2) is the unique modular definition of the holomorphic superpotential. The ambiguity can also be understood from the point of view of the $\mathcal{N}=2^*$ theory, where it corresponds to a freedom in defining different coordinates on the Coulomb branch. In fact, for gauge group SU(2), the additive

[3] In the case where N is a prime number, this result, as well as the corresponding formulae for the VEVs of $\mathcal{N}=1^*$ chiral condensates, appears in [8]. The general result was independently derived in [5].

shift described above corresponds precisely to the mismatch between the modular covariant Seiberg-Witten coordinate [13, 16] and the physical VEV $\langle \mathrm{tr}(\Phi_i^2)\rangle$. From this point of view it is also obvious that the ambiguity is vacuum independent. As we discuss extensively below, this reflects a more general phenomenon: operators which are uniquely defined in the $\mathcal{N}=4$ theory can mix with mass-dependent coefficients once the mass deformation is turned on.

The total number of massive vacua of the $\mathcal{N}=1^*$ theory is given by the sum over divisors of N, $\sum_{d|N} d$. In fact, this theory realizes every possible massive phase seen in 't Hooft's abstract classification of phases of $\mathrm{SU}(N)$ gauge theories with a $\mathbb{Z}_N$ symmetry [17]. The standard "Higgs" vacuum in which the gauge group is completely broken classically arises for $p=N, q=1$, while the "confining" vacua (for which classically $\Phi_i=0$), which are related to the vacua of the $\mathcal{N}=1$ pure SYM theory in an appropriate weak-coupling limit, arise for $p=1, q=N$. The "Higgs" vacuum corresponds to a condensation of electric charges and confinement of magnetic charges, while the "confining" vacua correspond to a condensation of magnetic (or dyonic) charges and a confinement of electric charges. For prime N there are no additional massive vacua. The weak coupling expansion of E_2 implies that vacua corresponding to integer values of p/q have a weak coupling expansion which can be interpreted as arising purely from instanton corrections; in other vacua additional non-perturbative effects, which may perhaps be associated with fractional instantons or merons, appear. Modular transformations act on the vacua by a non-trivial permutation, which follows from identifying the electric and/or magnetic charge of the particles which condense in each vacuum.

From (4.2) we can obtain the tensions of BPS-saturated domain walls between two different vacua, by computing the absolute values of superpotential differences between the vacua [18]. The superpotential difference between generic massive vacua is

$$\Delta W_{1,2} = \frac{N^2}{24}\, \tilde{m}_1\, \tilde{m}_2\, \tilde{m}_3 \left[p_1^2\, E_2\left(\frac{p_1}{q_1}\tau + \frac{k_1}{q_1}\right) - p_2^2\, E_2\left(\frac{p_2}{q_2}\tau + \frac{k_2}{q_2}\right)\right]. \tag{4.3}$$

One can check that this is a modular form of weight

$$3\left(-\frac{5}{6}, \frac{1}{6}\right) + (2,0) = \left(-\frac{1}{2}, \frac{1}{2}\right) \tag{4.4}$$

up to permutations of the vacua, so the domain wall tensions $T = |\Delta W|$ are modular invariant up to permutations, as they should be. In terms of the standard mass parameters the coefficient of these expressions involves $m_1\, m_2\, m_3/\lambda = \mathrm{Im}(\tau) m_1\, m_2\, m_3/4\pi N$ instead of $\tilde{m}_1\, \tilde{m}_2\, \tilde{m}_3$; the m_i's are the physical masses, and they are also the natural deformation parameters in SUGRA as described above.

The superpotential (4.2) also determines the values of the condensates of the chiral operators $\tilde{u}_2^{(i)} \equiv \mathrm{tr}(\tilde{\Phi}_i^2)$ and $S \equiv \mathrm{tr}(W_\alpha^2)$ in the various $\mathcal{N}=1^*$ vacua. This is a consequence of the usual identifications

$$\left\langle \tilde{u}_2^{(i)} \right\rangle = \frac{1}{N}\frac{\partial W}{\partial \tilde{m}_i}\,; \qquad \langle S\rangle = -16\,\pi\, i\frac{\partial W}{\partial \tau}\,. \tag{4.5}$$

In the normalization of (3.2), S is also a chiral superfield of the $\mathcal{N}=1^*$ theory, and its correlation functions should be holomorphic in τ and in the $\tilde{m}_i$. Assuming equal masses[4] and using $\tilde{m}_i = m_i/\lambda^{1/3}$, $\tilde{u}_2 = u_2/\lambda^{2/3}$, we see that

$$\langle u_2\rangle = \frac{N^2}{24}\, m^2 \left(E_2(\tau) - \frac{p}{q}\, E_2\left(\frac{p}{q}\tau + \frac{k}{q}\right)\right)\,, \tag{4.6}$$

where we have set $A(\tau, N) = 0$, thus providing a modular covariant definition of the holomorphic condensate. Via (2.2) this immediately leads to a modular invariant result for the VEV of $\mathcal{O}_2$,

$$\langle \mathcal{O}_2\rangle = \frac{N^2}{96\pi}\, \mathrm{Im}(\tau)\, m^2 \left[E_2(\tau) - \frac{p}{q}\, E_2\left(\frac{p}{q}\,\tau + \frac{k}{q}\right)\right]. \tag{4.7}$$

Up to permutations between the vacua of the $\mathcal{N}=1^*$ theory this operator is invariant under $\mathrm{SL}(2,\mathbb{Z})$, as expected.

From the definition (2.4), the operator $g_{\mathrm{YM}}^2\, Q^2\, \mathcal{O}_2$ of the $\mathcal{N}=4$ theory[5] is given by $\mathrm{tr}(\psi_4\,\psi_4) + 2\,\mathrm{tr}(\Phi_1[\Phi_2,\Phi_3]) + g_{\mathrm{YM}}^2\, NK$, the last term representing the anomalous contribution. The first term $\mathrm{tr}(\psi_4\,\psi_4)$ is the gluino condensate, which is the lowest component of the $\mathcal{N}=1$ chiral superfield $S = \mathrm{tr}(W_\alpha^2)$. The VEV of this operator is holomorphic in τ and can be obtained from the exact superpotential (4.2) using the second relation in (4.5).

Before the mass deformation the sum of the last two terms is $Q_{\mathcal{N}=1}$-exact and, therefore, cannot get a VEV. However, this is no longer true in the mass deformed theory. In fact, the VEV of these terms can be determined by considering the operator $\bar{Q}^2_{\mathcal{N}=1}\,\mathrm{tr}(\bar{\Phi}_1\,\Phi_1)$, which is classically proportional to $\mathrm{tr}(F_1^*\,\Phi_1)$. Quantum mechanically it is well-known [11, 12] that this is accompanied by the anomalous piece $-g_{\mathrm{YM}}^2\, NK$. The Q-exactness of this combination leads to a relation between the F-terms and the anomaly, of the form

$$2\,\langle \mathrm{tr}(\Phi_1 F_1^*)\rangle = -4\left\langle m_1\,\mathrm{tr}(\Phi_1^2)\right\rangle - 2\,\langle \mathrm{tr}(\Phi_1[\Phi_2,\Phi_3])\rangle = g_{\mathrm{YM}}^2\, N\,\langle K\rangle\,. \tag{4.8}$$

This relates the VEV of $\mathrm{tr}(\Phi_1^2)$ to the VEV of $2\,\mathrm{tr}(\Phi_1[\Phi_2,\Phi_3]) + g_{\mathrm{YM}}^2\, NK$ which is precisely the combination appearing in $Q^2\,\mathcal{O}_2$. Thus, we find

$$\begin{aligned}\left\langle Q^2\,\mathcal{O}_2\right\rangle &= \frac{1}{g_{\mathrm{YM}}^2}\left(\langle \mathrm{tr}(\psi_4\,\psi_4)\rangle - 4\left\langle m_1\,\mathrm{tr}(\Phi_1^2)\right\rangle\right)\\ &= -\frac{16\,\pi\, i}{g_{\mathrm{YM}}^2}\left(\frac{\partial W}{\partial\tau} - \frac{i}{\mathrm{Im}(\tau)}\, W\right).\end{aligned} \tag{4.9}$$

[4]It is easy to generalize our results to arbitrary masses just by using the global symmetries, which determine for example that $\langle \tilde{u}_2^{(1)}\rangle \propto m_2 m_3$ and $\langle S\rangle \propto m_1 m_2 m_3$.

[5]Note that Q here denotes the supercharges of the original $\mathcal{N}=4$ theory and not the $\mathcal{N}=1$ supercharge which is preserved after the mass deformation.

This is simply $-4\,i\,\mathrm{Im}(\tau)\,\tilde{m}^3$ multiplied by the modular covariant derivative[6] of the form $W/\tilde{m}^3$ of weight $(2,0)$. The expression (4.9) has the correct modular weight $(1/2,-1/2)$, thus providing a modular covariant (up to permutations) result for the VEV of $Q^2\,\mathcal{O}_2$

$$\begin{aligned}\left\langle Q^2\,\mathcal{O}_2\right\rangle &= -4\,i\,\mathrm{Im}(\tau)\,\tilde{m}^3 \frac{\mathcal{D}}{\mathcal{D}\tau}\left(\frac{W}{\tilde{m}^3}\right)\\ &= \frac{N^2}{24\,\pi\,i}\,(\mathrm{Im}(\tau))^2 m^3\left[E_2'(\tau)-\frac{p^2}{q^2}\,E_2'\left(\frac{p}{q}\,\tau+\frac{k}{q}\right)-\right.\\ &\qquad\left.-\frac{i}{\mathrm{Im}(\tau)}\left\{E_2(\tau)-\frac{p}{q}E_2\left(\frac{p}{q}\tau+\frac{k}{q}\right)\right\}\right]. \end{aligned} \tag{4.10}$$

The identification of the SUSY charge Q^2 with the modular covariant derivative appears to be a general result. It is familiar in the context of the $\mathcal{R}^4$ term in the type-IIB effective action (either on $\mathbb{R}^{10}$ [19] or $AdS_5\times S^5$ [20]), and it is inherited by correlation functions of chiral primary operators in the $\mathcal{N}=4$ theory via the AdS/CFT correspondence.

5. Operator mixing and a new modular symmetry

Before proceeding, we should reconsider the additive ambiguity which appears in the definition of the operators discussed in the previous section. Since the ambiguity is vacuum independent, it is convenient to describe it in terms of a mixing between the operator $\mathcal{O}_2$, as defined above, and $m^2\,\mathcal{I}$ where $\mathcal{I}$ is the identity operator. In particular, we may redefine $\mathcal{O}_2$ according to the operator equation

$$\mathcal{O}_2 \longrightarrow \tilde{\mathcal{O}}_2 = \mathcal{O}_2 + m^2\ \mathrm{Im}(\tau)\,f^{(2)}(\tau,\bar{\tau})\,\mathcal{I}\,, \tag{5.1}$$

where $f^{(2)}(\tau,\bar{\tau})$ is an arbitrary function of τ and $\bar{\tau}$ reflecting the fact that, apriori, $\tilde{\mathcal{O}}_2$ does not have any special holomorphy properties. Once again, it is convenient to restrict our attention to cases where $\tilde{\mathcal{O}}_2$ transforms with definite modular weight. This requires that $f^{(2)}(\tau,\bar{\tau})$ has modular weight $(2,0)$. However, unlike the holomorphic case considered above this is not prohibitive: there are an infinite number of possible choices of $f^{(2)}$ which preserve the modular properties of $\mathcal{O}_2$.

It turns out that exactly one choice of $f^{(2)}$ has a very special property which we now discuss. Specifically, if we set,

$$f^{(2)}(\tau,\bar{\tau}) = -\frac{N^2}{96\,\pi}\left(E_2\,(\tau)-\frac{3}{\pi\ \mathrm{Im}(\tau)}\right), \tag{5.2}$$

[6]The modular covariant derivative $\mathcal{D}/\mathcal{D}\tau$ of a $(w,\bar{w})$ form $\mathcal{O}$ is given by $d\mathcal{O}/d\tau - i\,w\,\mathcal{O}/2\,\mathrm{Im}(\tau)$, which in turn yields a modular form of weight $(w+2,\bar{w})$.

JHEP06(2000)026

we find

$$\left\langle \tilde{\mathcal{O}}_2 \right\rangle = N^2\, m^2 \left[\frac{1}{32\,\pi^2} - \frac{\mathrm{Im}(\tau)}{96\,\pi}\, \frac{p}{q}\, E_2\left(\frac{p}{q}\tau + \frac{k}{q} \right) \right]. \tag{5.3}$$

First, we can see that the expression (5.3) does not depend separately on τ, p, q and k, but just on the combination $\tilde{\tau} \equiv (p\,\tau + k)/q$. Moreover, we have

$$\left\langle \tilde{\mathcal{O}}_2 \right\rangle = N^2\, m^2 \left[\frac{1}{32\,\pi^2} - \frac{\mathrm{Im}(\tilde{\tau})}{96\,\pi}\, E_2\left(\tilde{\tau} \right) \right], \tag{5.4}$$

which shows that, in each vacuum, $\tilde{\mathcal{O}}_2$ is actually a modular form in the variable $\tilde{\tau}$ with weight[7] $(+1,-1)$.

A similar discussion applies to $Q^2\,\mathcal{O}_2$. As this operator has mass dimension three, it can mix with $m\,\mathcal{O}_2$ and $m^3\,\mathcal{I}$:

$$Q^2\,\tilde{\mathcal{O}}_2 = Q^2\,\mathcal{O}_2 + m\,\mathcal{O}_2\ \mathrm{Im}(\tau) g^{(2)}\left(\tau,\bar{\tau}\right) + m^3\ \mathrm{Im}(\tau)^2\, h^{(4)}\left(\tau,\bar{\tau}\right) \mathcal{I}\,. \tag{5.5}$$

To preserve covariance under S-duality transformations, $g^{(2)}$ and $h^{(4)}$ must be (non-holomorphic) modular forms of weight $(2,0)$ and $(4,0)$, respectively. Note that this mixing (as well as (5.1)) is consistent with the SU(4) R-symmetry of the $\mathcal{N}=4$ theory.

As in the case of $\mathcal{O}_2$ there is a unique definition of this operator which is modular in $\tilde{\tau}$. This definition uses $g^{(2)} = 0$ and

$$h^{(4)}(\tau,\bar{\tau}) = -4\,i\,\frac{\mathcal{D} f^{(2)}}{\mathcal{D}\,\tau} = -\frac{N^2}{24\,\pi\,i}\left(E_2'(\tau) - \frac{i\,E_2(\tau)}{\mathrm{Im}(\tau)} - \frac{3}{2\,\pi\,i\,(\mathrm{Im}(\tau))^2} \right), \tag{5.6}$$

giving

$$\begin{aligned} \left\langle Q^2 \tilde{\mathcal{O}}_2 \right\rangle = N^2 m^3 \Bigg[& -\frac{1}{32\,\pi^2} + \frac{\mathrm{Im}(\tau)}{48\,\pi}\,\frac{p}{q} E_2\left(\frac{p}{q}\tau + \frac{k}{q} \right) - \\ & - \frac{(\mathrm{Im}(\tau))^2}{48\,\pi\,i}\,\frac{p^2}{q^2}\,E_2'\left(\frac{p}{q}\tau + \frac{k}{q} \right) \Bigg]. \end{aligned} \tag{5.7}$$

Rewriting this explicitly in terms of $\tilde{\tau}$, using the relation $E_2'(\tau) = (\pi i/6)(E_2^2(\tau) - E_4(\tau))$, we have

$$\left\langle Q^2\,\tilde{\mathcal{O}}_2 \right\rangle = -\frac{32\,\pi^2}{N^2\,m} \left\langle \tilde{\mathcal{O}}_2 \right\rangle^2 + \frac{N^2\,m^3}{288} (\mathrm{Im}(\tilde{\tau}))^2\, E_4(\tilde{\tau})\,. \tag{5.8}$$

This definition is also consistent with the identification of the supercharge Q^2 with $\mathrm{Im}(\tau)\,\mathcal{D}_\tau = \mathrm{Im}(\tilde{\tau})\,\mathcal{D}_{\tilde{\tau}}$. From now on, $\tilde{\mathcal{O}}_2$ and $Q^2\tilde{\mathcal{O}}_2$ will denote the specific expressions (5.4) and (5.8), which are the unique definitions of these operators which transform with definite weight under modular transformations in $\tilde{\tau}$. We will now argue that this corresponds to an interesting new symmetry of the theory which has not been noted previously.

[7] This modular weight in $\tilde{\tau}$ differs from the corresponding weight under ordinary S-duality transformations in τ simply because we have chosen to assign zero weight to the masses under the former symmetry, while they have non-trivial weights under the latter.

JHEP06(2000)026

To understand the origin of this symmetry it is useful to start by considering the corresponding theory with eight supercharges, denoted above by $\mathcal{N}=2^*$. The low-energy effective lagrangian for this theory on its Coulomb branch is determined by the hyper-elliptic curve given by Donagi and Witten [9]. For gauge group SU(N) the curve can be thought of as a branched N-fold cover of the standard torus, $T(\tau)$, the latter being specified by the complex equation $y^2=(x-e_1(\tau))(x-e_2(\tau))(x-e_3(\tau))$. The Coulomb branch contains singular submanifolds where some of the cycles of the curve degenerate and one or more BPS states become massless. Softly breaking $\mathcal{N}=2$ SUSY down to an $\mathcal{N}=1$ subalgebra by introducing a mass for the scalar field in the $\mathcal{N}=2$ vector multiplet lifts the Coulomb branch except at isolated singular points. More precisely, the massive vacua of the $\mathcal{N}=1^*$ theory correspond to points in moduli space where a maximal number of cycles degenerate. In the more familiar case of $\mathcal{N}=2$ SUSY Yang-Mills, the relevant singular curve for each $\mathcal{N}=1$ vacuum is a sphere. In the present case, as explained in [9], the maximally degenerate curve is an unbranched N-fold cover of the standard torus $T(\tau)$. Such N-fold covers are classified by three non-negative integers p, q and $k<q$, with $p\,q=N$, and they are themselves complex tori $T(\tilde{\tau})$ with $\tilde{\tau}=(p\,\tau+k)/q$ as above. As usual, the low-energy effective lagrangian depends only on the complex structure of the curve. It must therefore be invariant under modular transformations acting on $\tilde{\tau}$. As the chiral condensates of the $\mathcal{N}=1^*$ theory are also determined purely from the complex structure of the curve, they must also exhibit this symmetry. In the following we will refer to this symmetry as $\tilde{\mathrm{S}}$-duality.

Several features of $\tilde{\mathrm{S}}$-duality are noteworthy. As we discuss below, it is almost certainly not an exact symmetry of the $\mathcal{N}=1^*$ theory, but only of the chiral sector of the theory which is controlled by the Donagi-Witten curves. Even this point requires further qualification because the chiral sector usually denotes the ring of chiral operators of an $\mathcal{N}=1$ theory, like those discussed in section 4, whose VEVs depend holomorphically on τ. Indeed one usually refers to this as the holomorphic sector of the theory. The problem is illustrated by the holomorphic formula derived above,

$$\langle u_2\rangle=\frac{N^2}{24}\,m^2\left(E_2(\tau)-\frac{p}{q}\,E_2\left(\frac{p}{q}\tau+\frac{k}{q}\right)\right). \tag{5.9}$$

This formula is not modular in $\tilde{\tau}$ for two reasons. First, it depends on τ as well as on $\tilde{\tau}$. This can easily be fixed by a holomorphic vacuum-independent redefinition which simply subtracts off the first term. More importantly, (5.9) is not modular in $\tilde{\tau}$ because of the anomalous transformation law of E_2 mentioned above. This is harder to rectify. In fact, the only way that $\tilde{\mathrm{S}}$-duality can be restored is by a *non-holomorphic* additive redefinition, which effectively replaces $E_2(\tilde{\tau})$ by $\hat{E}_2(\tilde{\tau})=E_2(\tilde{\tau})-3/\pi\ \mathrm{Im}(\tilde{\tau})$ (and yields only a *vacuum-independent* shift of $\langle u_2\rangle$). Thus, the new symmetry is only manifest after choosing a very particular non-holomorphic mixing of the chiral operators, as we did above. This feature is very reminiscent of the holomorphic anomaly discussed in [21].

Unlike conventional S-duality which generically permutes the vacua, the new modular symmetry is a symmetry of each vacuum. In vacuum states which are invariant under ordinary S-duality (this is the case when $p/q = 1$ and $k = 0$), the two dualities coincide (up to the fact that we assigned different modular weights to the masses). One particularly interesting case is that of the confining vacuum with $p = 1$, $q = N$ and $k = 0$. In terms of the 't Hooft coupling $\lambda = g_{\rm YM}^2 N$, we have (for zero theta angle) $\tilde{\tau} = 4\pi i/\lambda$. Thus, in this vacuum, one generator of $\tilde{\rm S}$-duality is an inversion of the 't Hooft coupling: $\lambda \to (4\pi)^2/\lambda$. As the 't Hooft coupling corresponds to the radius of the geometry in units of $\sqrt{\alpha'}$, this transformation corresponds to a novel kind of T-duality or mirror symmetry of the IIB background (although we emphasize again this only applies to the chiral sector of the theory). As noted in [8], this relates the regime of large λ, where the IIB side of the AdS/CFT duality is tractable, to the physically interesting regime of small λ, where the field theory becomes weakly coupled at short distances. Note that the $\mathcal{N} = 1^*$ theory flows to the $\mathcal{N} = 4$ SYM theory in the UV. This theory is weakly interacting at small λ, but is believed to have quantitatively different properties at large λ. This makes it clear that the $\tilde{\rm S}$-duality is not a symmetry of the $\mathcal{N} = 1^*$ or $\mathcal{N} = 2^*$ theory at all length scales. This is consistent with the above discussion, which suggests that it is a symmetry of the $\mathcal{N} = 2^*$ theory in the IR, or of a particular chiral sector of the $\mathcal{N} = 1^*$ theory.

6. Comparison with the type-IIB string theory dual

The AdS/CFT correspondence provides an unambiguous relationship between the $\mathcal{N} = 4$ field theory chiral operators and type-IIB supergravity fields. However, as we saw in the previous section, once we turn on the mass deformations, the definitions of the operators on the field theory side becomes ambiguous due to the possibility of non-trivial operator mixing. We expect a similar ambiguity to occur on the type-IIB side. Importantly, a linear redefinition of the fields appearing in the SUGRA equations of motion does not (by definition) refer to any specific background. This reflects the fact (discussed above) that operator mixing leads to a vacuum independent redefinition of the condensates. With this in mind, the main prediction of the field theory analysis in the previous section is that there exists a (unique) definition of the operators $\mathcal{O}_2$ and $Q^2\,\mathcal{O}_2$ such that they transform with definite modular weights under $\tilde{\rm S}$-duality transformations in each vacuum. In this section we will compare this prediction against the SUGRA results of Polchinski and Strassler [5]. As the type-IIB results are only available in certain regimes of parameter space we certainly will not be able to provide a direct demonstration that $\tilde{\rm S}$-duality is present (in the

JHEP06(2000)026

same sense as in the field theory discussion above) on the string theory side of the correspondence. However, we will be able to perform several non-trivial checks of this hypothesis.

According to the results of [5], many of the vacua of the $\mathcal{N}=1^*$ theory can be approximately described in type-IIB string theory via supergravity on asymptotically $AdS_5\times S^5$ spacetimes with sets of 5-branes arranged at various AdS radii. The mass perturbation appears as a non-normalizable 3-form field. The 5-brane sources with world volume $\mathbb{R}^4\times S^2$ are wrapped at various angles around equators of the S^5 — a configuration that is rendered dynamically stable by the D3-brane charges of the 5-branes. Massive $\mathcal{N}=1^*$ vacua with $N=p\,q$ may be described by supergravity solutions corresponding to q D5-branes each carrying p units of D3-brane charge when $1/N\ll g^2_{\mathrm{YM}}\ll N/q^2$, or equivalently $1\ll \mathrm{Im}(p\,\tau/q)\ll p^2$, while they are appropriately described by p NS5-branes each carrying q units of D3-brane charge when $1\ll \mathrm{Im}(-q/p\,\tau)\ll q^2$.

The SUGRA results for the chiral condensates, which we denote as $\langle\mathcal{O}_2\rangle_{\mathrm{SG}}$ and $\langle Q^2\,\mathcal{O}_2\rangle_{\mathrm{SG}}$, can be read off from the solutions for the metric and the 3-form G_3 (respectively) in [5]. We note that the normalizations of [5] differ by factors of g_s from the SUGRA fields corresponding to $\mathcal{O}_2$ and $Q^2\,\mathcal{O}_2$ whose kinetic terms are of order one, and that we need to express the results of [5] in terms of the AdS curvature radius (rather than the string scale) to match with our normalizations. Translating the results of [5] to coincide with our normalizations, we find in the $N=p\,q$ vacuum with $k=0$

$$\begin{aligned}
&\langle\mathcal{O}_2\rangle_{\mathrm{SG}}\propto N^3\,m^2\frac{\mathrm{Im}(\tau)}{q^2}\,, &&\big\langle Q^2\,\mathcal{O}_2\big\rangle_{\mathrm{SG}}\propto N^3\,m^3\frac{\mathrm{Im}(\tau)}{q^2}\,,\\
& &&\text{for } 1\ll \mathrm{Im}(p\,\tau/q)\ll p^2\,;\\
&\langle\mathcal{O}_2\rangle_{\mathrm{SG}}\propto N^3\,m^2\frac{\mathrm{Im}(\tau)}{p^2\tau^2}\,, &&\big\langle Q^2\,\mathcal{O}_2\big\rangle_{\mathrm{SG}}\propto N^3\,m^3\frac{\mathrm{Im}(\tau)\bar\tau}{p^2\tau^3}\,,\\
& &&\text{for } 1\ll \mathrm{Im}(-q/p\tau)\ll q^2\,.
\end{aligned}\tag{6.1}$$

Note that the results in the second line differ by a phase from the result presented in [5], but we believe that the phase in (6.1) is the correct one.

These expressions depend on τ, p and q only in the combination $\tilde\tau$. The dependence on $\tilde\tau$ alone is in fact a property not just of these VEVs but of the full supergravity solutions of [5] (though it does not seem to be a property of the full string theory solutions when string loop and α' corrections are taken into account). Whenever two configurations with the same $\tilde\tau$ both have good descriptions in terms of branes in [5], the corresponding supergravity solutions are the same (including having branes at the same place, which produce the same supergravity fields around them). Clearly, the fact that the SUGRA condensates depend exclusively on $\tilde\tau$ is in line with the field theory predictions of the previous section. In particular, exclusive

JHEP06(2000)026

dependence on $\tilde\tau$ is clearly a prerequisite for modular covariance in this parameter. Strictly speaking, the field theory prediction required only that the *exact* expressions on the string theory side should depend exclusively on $\tilde\tau$ (modulo field redefinitions). However, in the regime where (6.1) is valid supergravity is a good approximation and therefore the supergravity results should themselves depend exclusively on $\tilde\tau$, as we indeed find from (6.1).

Another non-trivial check comes from noting that, at least for the vacua in which both p and q grow with positive powers of N (subject to $p\,q = N$), there is an overlap between the range of values of $\tilde\tau$ for which the first line is valid and the range of values of $\tilde\tau$ for which the second line is valid after an $\tilde{\mathrm{S}}$-duality transformation $\tilde\tau \to -1/\tilde\tau$. It is easy to check that the expressions appearing in the first and second lines are precisely related by such a transformation, using the modular weights $(+1,-1)$ and $(+2,-2)$ for $\mathcal{O}_2$ and $Q^2\,\mathcal{O}_2$ under $\tilde{\mathrm{S}}$-duality as in the previous section.

Actually, the agreement between the string theory and field theory results is much stronger than this. To make a more detailed comparison, we will use both the weak coupling expansion of $E_2(\tilde\tau)$,

$$E_2(\tilde\tau) = 1 - 24 \sum_{k=1}^{\infty} \frac{k\, e^{2\,\pi\, i\, k\, \tilde\tau}}{1 - e^{2\,\pi\, i\, k\, \tilde\tau}}\,, \tag{6.2}$$

and the "strong coupling" expansion for large $\mathrm{Im}(-1/\tilde\tau)$, of the form

$$E_2(\tilde\tau) \simeq \frac{1}{\tilde\tau^2} + \frac{6\,i}{\pi\,\tilde\tau} + \mathcal{O}(e^{-2\,i\,\pi/\tilde\tau})\,. \tag{6.3}$$

Using these identities we can see that throughout their domain of validity, the SUGRA expressions for the VEVs of both operators agree precisely with the unique $\tilde{\mathrm{S}}$-covariant field theory expressions derived in the previous sections. Thus, we have $\langle\mathcal{O}_2\rangle_{\mathrm{SG}} = \langle\tilde{O}_2\rangle$ and $\langle Q^2\,\mathcal{O}_2\rangle_{\mathrm{SG}} = \langle Q^2\,\tilde{O}_2\rangle$ whenever the string theory expressions are valid. As discussed above, the field theory results only require that these equalities should hold modulo field redefinitions or operator mixings. It would be interesting to understand the origin of the stronger result we have found. Note that the supergravity results of [5] generally do not agree with the naive (unmixed) VEVs that we computed in section 4.

So far we have only discussed the results on the string theory side in the supergravity regime. The field theory predictions imply that, with suitable background independent field redefinitions, the *exact* string theory results for the condensates of $\mathcal{O}_2$ and $Q^2\,\mathcal{O}_2$ should agree with the field theory expressions (5.3) and (5.7). Of course, it is not possible to test this directly, due to our ignorance about string theory in backgrounds with non-zero RR flux. However, certain qualitative observations can be made in support of this proposition. In particular, in each vacuum, the field theory results for large values of $\mathrm{Im}(\tilde\tau)$ have an expansion in powers of $\exp(2\,\pi\,i\,\tilde\tau)$

by virtue of (6.2), while for large values of $\mathrm{Im}(-1/\tilde{\tau})$ we have the $\tilde{\mathrm{S}}$-dual expansion in powers of $\exp(-2\,\pi\,i/\tilde{\tau})$. It is natural to try to identify these exponentially suppressed terms with the contributions of string theory instantons on the type-IIB side. Such an identification was given in [5] and we will expand on it somewhat in the following.

At large $\mathrm{Im}(\tilde{\tau})$, the corresponding weakly-coupled type-IIB background includes q D5-branes with world volume $S^2 \times \mathbb{R}^4$, where the radius of the sphere is proportional to p. By a slight extension of the results given in [5], one may show that the action of a single D-string worldsheet wrapped on the S^2 is precisely $2\,\pi\,i\,\tilde{\tau}$. Provided it preserves half of the unbroken supersymmetry (i.e. two supercharges), this instanton configuration will contribute terms of order $\exp(2\pi i\tilde{\tau})$ to the lowest terms in the derivative expansion of the type-IIB effective action, which determine the large-distance behavior of the SUGRA fields and thereby determine the condensates. The field theory results suggest that multiple wrappings of the sphere should also contribute. The fact that such configurations exist is plausibly related to the fact that, at least on $\mathbb{R}^{10}$, any number of D1-branes and D5-branes can form a stable bound state at threshold which saturates the BPS bound and therefore preserves half the supersymmetry. This state then contributes as an instanton after compactification on S^2. As there are no analogous bound states of (m,n)-strings (with $(m,n) \neq (0,1)$) with D5-branes, this also suggests an explanation of why only configurations involving wrapped D-strings contribute.

In the case where $\mathrm{Im}(\tilde{\tau}) \ll 1$ with $k=0$, the type-IIB background includes p NS5-branes wrapped on $\mathbb{R}^4 \times S^2$ with radius proportional to q. By similar arguments, wrapping the type-IIB string worldsheet on S^2 yields an instanton with action $2\pi i/\tilde{\tau}$ which contributes terms of order $\exp(-2\pi i/\tilde{\tau})$ to the condensates as expected. The corresponding statements about the existence of bound states of a fundamental string with an NS5-brane are related by S-duality to the D1/D5 case discussed above. Finally, in vacua with $k > 0$, the instanton expansion comes from (p,k)-strings wrapped on the S^2 factor of a (p,k) fivebrane world volume.

Of course, there are many other possible sources of corrections to the classical supergravity results. For example, we expect the D-instantons of the type-IIB theory to contribute terms of order $\exp(2\pi i\tau)$ in the weak coupling limit $\mathrm{Im}(\tau) \to \infty$. Indeed, such terms are present in the naive, unmixed, field theory expressions for the condensates, (4.7) and (4.10). Also, there is no obvious reason why perturbative corrections, both in the string coupling and in α', cannot contribute. However, the prediction is not that these contributions are absent but rather that they can be removed by a background independent redefinition of the SUGRA (string theory) fields. In the case of D-instantons, the fact that their action is independent of the integers p, q and k which characterize the vacuum means that this appears to be consistent. In fact, the operator mixing on the field theory side has precisely the effect of removing these terms. In contrast, the string instantons discussed above have an action which depends on p, q and k and thus could not be removed by any such redefinition. On the field theory side, the absence of perturbative corrections in our formulae ultimately comes from the existence of a holomorphic superpotential. On the string theory side we would also like to understand why perturbative corrections in the string coupling are absent modulo field redefinitions. Our results suggest that there are quantities which are 'almost holomorphic' on the type-IIB side but suffer from a mild holomorphic anomaly which restores $\tilde{\mathrm{S}}$-duality, as in [21].

Finally, we note that the appearance of $\tilde{\mathrm{S}}$-duality on the type-IIB side of the correspondence is somewhat mysterious as there is no torus apparent in the geometry. On the other hand, the $\mathcal{N} = 2^*$ theory can be realized on a network of intersecting branes in type-IIA string theory [22]. After lifting to M-theory, the theory is realized as a compactification of the six-dimensional $(2,0)$ superconformal theory which lives on the M5-brane. The compactification manifold is precisely the relevant Donagi-Witten curve. At the singular points in the moduli space this is a torus with complex structure $\tilde{\tau}$. Hence, $\tilde{\mathrm{S}}$-duality has a natural geometrical realization in this construction. This is very similar to the T-dual realization of type-IIB S-duality in M-theory on $T^2 \times \mathbb{R}^9$. Presumably, soft breaking to the $\mathcal{N} = 1^*$ theory could be accomplished along the lines of [23, 24]. The relation between the resulting type-IIA/M-theory set-up and the type-IIB construction of the same theory discussed above is not obvious. As above, this is very suggestive of T-duality, but it would be interesting to make this more precise.

7. Large-N limits of Higgs and confining vacua

In the rest of this paper we will attempt to ascertain which vacua of the $\mathcal{N} = 1^*$ field theory could be described by supergravity solutions without 5-brane sources, of the type analyzed in [6]. For such vacua, in the large-N limit the dual string theory solution should converge to a fixed metric in some finite region near the boundary (say, for the radial coordinate bigger than a fixed number times the AdS radius). There are two related ways to analyze this question. One approach is to look at the scaling of the VEVs of chiral operators in a large-N limit (there are two such limits as we discuss below). The existence of a fixed SUGRA description in this limit would mean that correlation functions and VEVs should scale as N^2 since the classical supergravity action scales as N^2 in our normalization. In particular we would expect $\langle \mathcal{O}_2 \rangle_{\mathrm{SG}}$ and $\langle Q^2\, \mathcal{O}_2 \rangle_{\mathrm{SG}}$ to scale this way. An alternative approach is to look at the solutions of [5] for the string theory duals of $\mathcal{N} = 1^*$ vacua and ask when they converge to a fixed supergravity background in the large-N limit. Since the positions of the 5-branes in these solutions are linked to the VEVs of chiral operators, the two approaches are in fact related.

There are two different types of large-N limits that could correspond to a SUGRA background. One is the large-N limit with λ fixed and large, which corresponds to a supergravity background in which one can have both a string perturbation expansion and an α' expansion, corresponding to first performing an expansion in $1/N$ (for constant λ) and then in $1/\sqrt{\lambda}$. The other is a large-N limit with fixed coupling, in which one still has a $1/N$ expansion. In both cases we expect all correlation functions to scale as N^2.

Let us start by looking at the standard Higgs and confining vacua of the $\mathcal{N}=1^*$ theory. In these vacua, the position of the 5-branes in the solution of [5] (in AdS units) scales as $\sqrt{N}$ in the large-N limit with fixed coupling, so the gravitational background is not fixed in this limit. In the large-N limit with fixed (and large) λ, the position of the D5-branes in the Higgs vacuum scales as $N/\sqrt{\lambda}$, while the position of the NS5-branes in the confining vacuum scales as $\sqrt{\lambda}$. Thus, it is clear just from looking at the 5-branes that these vacua cannot have supergravity duals, since the position of the 5-branes (which are sources for the supergravity fields) is not fixed in the large-N limit, (nor in the large N, large λ limit). We will see that the same result follows from looking at the large-N limit of the chiral condensates.

Using our results above (using either the field theory or the results of [5], since both agree whenever supergravity might be valid) we can easily compute the behavior of the chiral condensates in the large-N limit. Let us start with the large N, fixed (large) λ limit. We find in the Higgs vacuum ($p=N, q=1$)

$$\langle \mathcal{O}_2 \rangle_{\mathrm{SG}} \propto \frac{N^4\, m^2}{\lambda}\,, \tag{7.1}$$

while in the k-th confining vacuum

$$\langle \mathcal{O}_2 \rangle_{\mathrm{SG}} \propto N^2\, m^2\, \lambda \tag{7.2}$$

(with an additional term scaling as $k\, N\, m^2\, \lambda^2$ if we keep k constant and non zero in the large-N limit). Note that m is normalized so that it couples directly to a descendant of $\mathcal{O}_2$, so we expect it to be exactly the natural parameter from the supergravity point of view. Both expressions are too large and have the wrong N or λ scaling to appear naturally in the constant λ supergravity limit. Similarly, the vacuum expectation value of $Q^2\, \mathcal{O}_2$ in the zeroth confining vacuum includes terms scaling as $N^2\, m^3\, \lambda$, while in the Higgs vacuum it scales as $N^4\, m^3/\lambda$, both of which have the wrong N or λ scaling to appear in the constant λ limit in supergravity. A related observation made in [8] is that the tension of a BPS domain wall which interpolates between Higgs and confining vacua scales like N^4, which is hard to explain in terms of a configuration interpolating between fixed SUGRA backgrounds. In contrast, this behavior is correctly reproduced by the construction of domain walls as five-brane junctions given in [5]. Another puzzle raised in [8] is that, in the context of the singular solutions of [6], there is no obvious explanation for the worldsheet instantons contributing to quantities in the confining vacuum. As discussed in the previous section, this also has a natural resolution in the construction of [5].

In principle, even if the large-λ limit is not described by SUGRA, it could still describe the limit of large N and constant τ, which could correspond to a supergravity background at some fixed value of the string coupling. (A weak coupling expansion will generally not be possible in such a vacuum; for example, these could be vacua involving NS 5-branes, for which string perturbation theory breaks down.) As described above, the solutions of [5] suggest that this is not possible in the Higgs and confining vacua, and we can verify also that the chiral VEVs in these vacua are too large for SUGRA in this limit; both $\langle \mathcal{O}_2 \rangle_{\rm SG}$ and $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG}$ scale as N^3. In the Higgs vacuum we find $\langle \mathcal{O}_2 \rangle_{\rm SG} \simeq -N^3\, m^2\, {\rm Im}(\tau)/96\,\pi$ and $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG} \simeq N^3\, m^3\, {\rm Im}(\tau)/24\,\pi$, while in the zeroth confining vacuum (with $\theta_{\rm YM} = 0$) we find $\langle \mathcal{O}_2 \rangle_{\rm SG} \simeq N^3\, m^2/(96\,\pi\, {\rm Im}(\tau))$ and $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG} \simeq N^3\, m^3/(24\,\pi\, {\rm Im}(\tau))$. These expressions are consistent with the S-duality relation between these vacua (recall that S-duality also changes the phase of m).

Thus, we conclude that the Higgs and confining vacua are not dual to fixed supergravity backgrounds in the large-N limit (even if we lift the requirement of having a good string perturbation expansion around these backgrounds). The only dual description of these vacua is through the full solutions of [5].

8. Supergravity-like large-N limits

As we saw above, the Higgs and confining vacua are not dual to fixed supergravity backgrounds in the large-N limit; however, there is another large-N limit of massive vacua which could be dual to such a background (at least asymptotically). This is the limit of large N, when we take the coupling to be constant and look at sequences of vacua with p/q values converging to a fixed number in the large-N limit. For example, we can take $p/q = 2$ and look only at values of N of the form $N = p\,q = 2\,q^2$ for integer q's. If we also take k/q to be constant in the large-N limit (for example, we can take $k = 0$), we find in this limit that $\langle \tilde{\mathcal{O}}_2 \rangle$ and $\langle Q^2 \tilde{\mathcal{O}}_2 \rangle$ converge to expressions scaling as N^2 , consistent with a supergravity description. The expressions for the higher $\langle \mathcal{O}_k \rangle$'s which can be found in the appendix of [8] (without mixings) also appear to have the correct scaling in this limit.

In the stringy description of the massive vacua in [5], the main difference between these vacua and the confining and Higgs vacua is in the position of the 5-branes. In the large-N limit with constant p/q, the 5-branes are at a finite radial position in AdS units. For example, for ${\rm Im}(\tilde{\tau}) \gg 1$ where we can describe these vacua in terms of D5-branes, the radial position of the D5-branes in units of the AdS radius is a constant times $m\sqrt{{\rm Im}(\tilde{\tau})}$. Thus, these solutions converge to a fixed background of string theory in the large-N limit. In terms of the solutions of [5] the number of 5-branes grows in this limit but α' becomes smaller, since we are keeping the string coupling fixed, such that the effect of the 5-branes on the background remains the same. Because of this property it makes sense to compare this limit with the supergravity solutions for deformations such as those of [6], and we will do this below. Note that unlike $AdS_5 \times S^5$, the vacua we describe here only exist for specific values of N; e.g. if we take $p/q = 2$ then N has to be of the form $N = 2\,q^2$ for some integer q. However, this quantization is invisible in the supergravity limit (though it is obvious when these vacua are described in terms of branes [5]).

In [6], solutions of five-dimensional supergravity were described that correspond to mass deformations of the type we analyzed here (with equal masses). These solutions are all singular. However, since five-dimensional supergravity is believed to be a consistent truncation of type-IIB supergravity on $AdS_5 \times S^5$, it is believed that they can be extended into solutions of ten-dimensional supergravity, which would presumably still be singular. The authors of [6] looked only for solutions with $\langle \mathcal{O}_2 \rangle_{\rm SG} = 0$, and found such solutions both with $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG} = 0$ and with $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG} \neq 0$. The generic vacuum that we find in the limit discussed above has both $\langle \tilde{\mathcal{O}}_2 \rangle$ and $\langle Q^2\, \tilde{\mathcal{O}}_2 \rangle$ non zero, so it does not correspond to any of the vacua of [6]. However, for particular values of τ we can get $\langle \tilde{\mathcal{O}}_2 \rangle = 0$, and these values could correspond to some of the solutions of [6] (which necessarily exist for all τ, since there is no potential for the dilaton in the supergravity approximation). Using the conjectured form (5.4), (5.8) for the VEVs, we find that $\langle \tilde{\mathcal{O}}_2 \rangle = 0$ is solved by $\tilde{\tau} = e^{2\,\pi\, i/3}$ or values related to this by $\mathrm{SL}(2,\mathbb{Z})$. Plugging this value back into (5.8), we find that these solutions also satisfy $\langle Q^2\, \tilde{\mathcal{O}}_2 \rangle = 0$. Thus, we seem to have found candidate vacua that match the solutions of [6] with $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG} = 0$. Obviously this matching would only be valid in a regime where the solutions of [6] are not singular, otherwise these solutions must be corrected. In fact, in the cases where the analysis of [5] is valid, the solutions can acquire corrections (due to the presence of the 5-branes) even before the singularity is reached, but always a finite distance away in AdS units. We conclude that the singularity found by [6] in the solution with $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm SG} = 0$ should be resolvable in string theory, and that similar resolvable singular solutions should exist with non-zero $\langle \tilde{\mathcal{O}}_2 \rangle$ and $\langle Q^2\, \tilde{\mathcal{O}}_2 \rangle$ related by (5.8).

On the other hand, we find no solutions which in the large-N limit have $\langle \tilde{\mathcal{O}}_2 \rangle = 0$ and $\langle Q^2\, \tilde{\mathcal{O}}_2 \rangle$ non zero. Thus, it seems that the singularities found in solutions with this property in [6] would not be allowed in string theory, at least in asymptotically AdS spaces, since the field theory does not appear to have any corresponding vacua.

Our analysis is based on the expressions which we only know to be true in the large-N limit. For finite N we do not have good arguments in favor of supergravity expressions coinciding precisely with the definition of the field theory operators which is $\tilde{\mathrm{S}}$-duality covariant. We might actually get a larger family of solutions with four real parameters (τ, p/q and k/q), which degenerates into a two-parameter family of solutions in the large-N limit. If this is the case then in the full type-IIB string theory there will be solutions with $\langle \mathcal{O}_2 \rangle_{\rm IIB} = 0$ and $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm IIB}$ non zero, but $\langle Q^2\, \mathcal{O}_2 \rangle_{\rm IIB}$ in these solutions would have to grow slower than the standard supergravity scaling in the large-N limit, so that these solutions cannot correspond to those found in [6].

So far we have focused only on the massive vacua of the deformed $\mathcal{N}=4$ theory, even though for large N there are many more massless vacua (with massless photons and no mass gap). Unfortunately, the solutions to the equations of motion of (4.1) corresponding to generic massless vacua are not known, and therefore their analysis is much harder. It seems reasonable to expect that the operator VEVs in the massless vacua will be of the same order as those in the massive vacua, so that there may also be series of massless vacua that converge in the large-N limit to SUGRA solutions. In principle they could even converge to the same solutions, meaning that the singularity in these solutions should have more than one possible resolution in string theory. Examples of this are provided by $\mathrm{SL}(2,\mathbb{Z})$ invariant massless vacua which exist for certain values of N. For example, when N is of the form $N=(l^2-1)k^2/l^2$ for integers k and l (with k divisible by l), the superpotential (4.1) is extremized by $X_a=(i+j\,\tau)/k$, where i and j go over all pairs of integers from 0 to $k-1$ except those where both integers are divisible by l. It is easy to check that this vacuum is $\mathrm{SL}(2,\mathbb{Z})$ invariant. Another series of $\mathrm{SL}(2,\mathbb{Z})$-invariant massless vacua arises for $N=k(k+1)/2$ from the N-dimensional representation of SU(2) corresponding to blocks of size $(1,2,3,\ldots,k-1,k)$. Since this is the only representation which leads to a $\mathrm{U}(1)^{k-1}$ gauge theory at low energies this vacuum must also be $\mathrm{SL}(2,\mathbb{Z})$ invariant. In such $\mathrm{SL}(2,\mathbb{Z})$ invariant vacua we must have $W=0$ (since $W/\tilde{m}_1\,\tilde{m}_2\,\tilde{m}_3$ must be a holomorphic modular form of weight two, which does not exist), so they have $\langle\mathcal{O}_2\rangle=\langle Q^2\,\mathcal{O}_2\rangle=0$ for the unmixed operators in the field theory, just like the massive vacua with $p=q$ and $k=0$. The VEVs of the mixed operators in these vacua are thus given by (5.4) and (5.8), with $p=q$ and $k=0$. These vacua seem to also converge to the SUGRA solution found in [6] with $\langle Q^2\,\mathcal{O}_2\rangle_{\mathrm{SG}}=0$. The values of N which give rise to these massless vacua are of course different from those that give rise to the massive vacua with $p=q$, but in the large-N limit the solutions corresponding to these different vacua would look very similar (although the string theory resolution of the singularities is quite different, as in [5]).

What do these results teach us about the resolution of the singularities appearing in the solutions of [6]? Assuming that a complete analysis of the massless vacua does not change our conclusions, it appears that the singularity appearing in the $\langle Q^2\,\mathcal{O}_2\rangle_{\mathrm{SG}}=0$ solutions should be resolvable while the others should not. In other words, there should be no solution of string theory in asymptotically AdS space which converges (for large enough radial coordinates) to the singular solution with $\langle Q^2\mathcal{O}_2\rangle_{\mathrm{SG}}\neq 0$. There seems to be no obvious way of distinguishing the resolvable singularities from the others directly. Of course, we expect that whenever a singularity can be resolved it will be in terms of some brane configuration as in [5], so the claim here is that no brane configuration can resolve most of the singularities of [6] in the sense described above.

Note that all the singularities of [6] obey the criteria of Gubser [25] for "good" singularities, so we seem to have an example of "good" singularities which are still disallowed. In this case it seems that the finite temperature criterion of [25] should not be relevant, since we would not expect the solution of a particular vacuum to have a finite temperature generalization; rather, at a temperature of order m (where we might expect to see a smooth horizon), the field theory is presumably in some state which is a superposition of the different vacua (and of other states).

It is not clear if this analysis teaches us something about resolving singularities with continuous parameters in flat space, such as those that appeared in [26, 27, 28]. There is obviously no direct relation since we are discussing ten-dimensional singularities in asymptotically AdS spaces, while they discuss five-dimensional singularities in flat space. However, our results suggest that generically it would not be possible to resolve such singularities in string theory.

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JHEP06(2000)026

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